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CORRECTION

Correction: Transport Inequalities on Euclidean Spaces for Non-Euclidean Metrics

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Proposition 6.1 of the paper is not correct, a fact which impacts a number of statements and conclusions. The note provides the necessary adjustments for the statement and the main corollaries to hold true, including Theorem 1.1. Basically, all the results are still true up to some logarithmic corrections.

The paper under correction is referenced below as [1], and all the notation are taken from it, although a few basic objects and definitions are recalled for convenience. Further results and generalizations of the Fourier analytic bounds obtained in [1] can be stated in terms of Zolotarev distances [3].

1 General Fourier Analytic Bounds

With any two probability measures μ and ν on the d-dimensional torus $Q^d = (-\pi, \pi]^d$, we associate their Fourier-Stieltjes transforms

$$f_\mu(m) = \int_{Q^d} e^{im\cdot x}\, d\mu(x), \quad f_\nu(m) = \int_{Q^d} e^{im\cdot x}\, d\nu(x), \quad m \in \mathbb{Z}^d.$$

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Proposition 6.1 in [1] asserts that

$$\widetilde{\mathrm{W}}_\omega(\mu,\nu) \le \sqrt{d}\,\Big(\sum_{m\neq 0} \omega^2\Big(\frac{\pi}{|m|}\Big)\,|f_\mu(m)-f_\nu(m)|^2\Big)^{1/2}. \tag{1}$$

Here $\omega : [0,\infty) \to [0,\infty)$ is an arbitrary modulus of continuity (a subadditive continuous function such that $\omega(0)=0$, $\omega(\delta)>0$ for $\delta>0$) and $\widetilde{\mathrm{W}}_\omega$ denotes the transport (Kantorovich) distance with respect to the metric

$$\tilde{\rho}(x,y) = \omega(\|x-y\|), \quad x,y\in Q^d,$$

where $\|z\|$ denotes the shortest Euclidean distance from a point z to the lattice $2\pi\mathbb{Z}^d$. Let us recall that

$$\widetilde{\mathrm{W}}_\omega(\mu,\nu) = \inf_\lambda \int_{Q^d}\int_{Q^d} \tilde{\rho}(x,y)\,d\lambda(x,y)$$

where the infimum runs over all probability measures λ on $Q^d\times Q^d$ with marginals μ and ν.

Although the inequality (1) is true for the standard modulus of continuity $\omega(\delta)=\delta$ (even with a dimension-free coefficient, cf. [2]), in the general case including $\omega(\delta)=\delta^\alpha$, $0<\alpha<1$, it needs to be corrected. This issue is deeply connected with embedding problems of fractional Sobolev spaces among which are the Lipschitz classes Lip(α), cf. [4].

The relation (1) will be saved if we put an additional logarithmically growing factor inside the sum on the right-hand side. For a precise statement, fix an arbitrary non-decreasing function $q:[1,\infty)\to(0,\infty)$ such that

$$C_q^2 = \sum_{k=0}^{\infty} \frac{1}{q(2^k)} < \infty \quad (C_q>0). \tag{2}$$

Proposition 6.1 (Corrected version) *Given two probability measures μ and ν on Q^d,*

$$\widetilde{\mathrm{W}}_\omega(\mu,\nu) \le C_q\sqrt{d}\,\Big(\sum_{m\neq 0} q(|m|)\,\omega^2\Big(\frac{\pi}{|m|}\Big)\,|f_\mu(m)-f_\nu(m)|^2\Big)^{1/2}. \tag{3}$$

For example, the choice $q(x)=\log^{1+\varepsilon}(2x)$ with a parameter $\varepsilon>0$ leads to

$$\widetilde{\mathrm{W}}_\omega(\mu,\nu) \le C_\varepsilon\sqrt{d}\,\Big(\sum_{m\neq 0} \log^{1+\varepsilon}(2|m|)\,\omega^2\Big(\frac{\pi}{|m|}\Big)\,|f_\mu(m)-f_\nu(m)|^2\Big)^{1/2}$$

with constant C_ε depending on ε only.

Proof At the end of proof of Proposition 6.1 in [1] we derived the upper bound

$$\sum_{2^{k-1}\le|m|<2^k} |a_m|\,|f_\mu(m) - f_\nu(m)| \le \sqrt{d}\,\omega(\pi\,2^{-k})\,b_k$$

with

$$b_k = \Big(\sum_{2^{k-1}\le|m|<2^k} |f_\mu(m) - f_\nu(m)|^2\Big)^{1/2}.$$

Only the very last step has to be corrected. Performing summation over all $k \ge 1$ and introducing the q factor, Cauchy's inequality yields

$$\Big(\sum_{m\neq 0} |a_m|\,|f_\mu(m) - f_\nu(m)|\Big)^2 \le d\,\sum_{k=1}^{\infty}\frac{1}{q(2^{k-1})}\,\sum_{k=1}^{\infty} q(2^{k-1})\,\omega^2(\pi\,2^{-k})\,b_k^2.$$

Here the first sum is equal to the constant C_q^2 in (2). By monotonicity, $q(2^{k-1}) \le q(|m|)$ for $2^{k-1} \le |m| < 2^k$. Hence the second sum does not exceed the sum in (3). □

Another version of Proposition 6.1 from [1] was given in Proposition 6.3. With the same argument, it should be corrected to the form

$$\widetilde{\mathrm{W}}_\omega(\mu,\nu) \le 2C_q\,\Big(\sum_{m\neq 0} q(|m|)\,\omega^2\Big(\frac{\pi\sqrt{d/2}}{|m|}\Big)\,|f_\mu(m) - f_\nu(m)|^2\Big)^{1/2}. \tag{4}$$

Several general bounds in [1], consequences of the preceding and related to the smoothing operations, should be corrected accordingly by adding the factor $q(|m|)$ inside the sums and the coefficient C_q in front. In particular, using again a non-decreasing function q satisfying (2), the main result, Theorem 1.1, should read as follows. The transport distance W_ω in this statement (and below) is defined similarly to $\widetilde{\mathrm{W}}_\omega$ with respect to the metric $\rho(x,y) = \tilde\rho(x,y) = \omega(|x-y|)$ on the cube $[0,\pi]^d$.

Theorem 1.1 (Corrected version) *Given two probability measures μ and ν on $[0,\pi]^d$, for any modulus of continuity ω and any $t > 0$,*

$$\mathrm{W}_\omega(\mu,\nu) \le C_q\sqrt{d}\,\Big(\sum_{m\neq 0} q(|m|)\,\omega^2\Big(\frac{\pi}{|m|}\Big)\,e^{-t|m|^2}\,|f_\mu(m) - f_\nu(m)|^2\Big)^{1/2} + 6\,\omega(\sqrt{dt}). \tag{5}$$

The proof of Theorem 1.1 follows the lines developed in Sect. 7 of the original paper, together with the new version of Proposition 6.1. Clearly, the statement of Proposition 7.1 therein has to be modified accordingly incorporating the additional weight q. Namely, the inequality (7.4) should take the form

$$\mathrm{W}_\omega(\mu,\nu) \le C_q\sqrt{d}\,\Big(\sum_{m\neq 0} q(|m|)\,\omega^2\Big(\frac{\pi}{|m|}\Big)\,|f_\mu(m)-f_\nu(m)|^2\,|h(m)|^2\Big)^{1/2} + 6\,\omega(\mathbb{E}(|H|)),$$

where h is the characteristic function of a random vector H in $\mathbb{R}^d$, and μ and ν are probability measures supported on $[0,\pi]^d$. Remark 7.2 is modified similarly.

The statements in [1] involving other choices of smoothing probability distributions, such as the ones having compactly supported characteristic functions, should be modified similarly. For example, as a direct consequence of (3), inequality (1.6) in [1] should be replaced by, for every $T>0$,

$$\mathrm{W}_\omega(\mu,\nu) \le C_q\sqrt{d}\,\Big(\sum_{1\le\|m\|_\infty\le T} q(|m|)\,\omega^2\Big(\frac{\pi}{|m|}\Big)\,|f_\mu(m)-f_\nu(m)|^2\Big)^{1/2} + 6\,\omega\Big(\frac{\sqrt{12d}}{T}\Big) \tag{6}$$

where $\|m\|_\infty = \max(|m_1|,\dots,|m_d|)$ for $m=(m_1,\dots,m_d)\in\mathbb{Z}^d$.

It is possible to derive a version of (6) without the q-factor but up to some additional logarithmic factor of T. Indeed, in view of the range of summation, it is sufficient to require that the function $q(x)$ be defined in the interval $1\le x\le T\sqrt{d}$. For simplification, one may use $q(|m|)\le q(T\sqrt{d})$. Moreover, the quantity $C_q^2\,q(T\sqrt{d})$ is minimized when all $q(2^k)$ are equal to each other for $2^k\le T\sqrt{d}$. Since this inequality is fulfilled for at most $1+\log_2(T\sqrt{d})$ values of k, (6) yields

$$\mathrm{W}_\omega(\mu,\nu) \le \sqrt{d\,\log_2(2T\sqrt{d})}\,\Big(\sum_{1\le\|m\|_\infty\le T} \omega^2\Big(\frac{\pi}{|m|}\Big)\,|f_\mu(m)-f_\nu(m)|^2\Big)^{1/2} + 6\,\omega\Big(\frac{\sqrt{12d}}{T}\Big). \tag{7}$$

Thus, with respect to (1.6) in [1], there is an additional factor of order $\sqrt{\log T}$ in front of the sum on the right-hand side.

2 Empirical Measures

Theorem 1.1 with a general q-factor can be applied to empirical measures

$$\mu_n = \frac{1}{n}\sum_{k=1}^n \delta_{X_k}, \quad \nu_n = \frac{1}{n}\sum_{k=1}^n \delta_{Y_k}$$

associated to random vectors $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ with values in $[0, \pi]^d$. The inequality (5) yields the corresponding correction of Proposition 2.1 in [1].

Proposition 2.1 (Corrected version) *Suppose that the couples (X_k, Y_k) are pairwise independent and that X_k and Y_k have equal distributions for every $k \le n$. For any $t > 0$,*

$$c\,\mathbb{E}\,(\mathrm{W}_\omega(\mu_n, \nu_n)) \le C_q \frac{\sqrt{d}}{\sqrt{n}} \bigg(\sum_{m \neq 0} q(|m|)\, \omega^2\Big(\frac{\pi}{|m|}\Big)\, e^{-t|m|^2} \bigg)^{1/2} + \omega(\sqrt{dt}) \tag{8}$$

with an absolute constant $c > 0$. Moreover, if all X_k, Y_l are independent, a similar inequality also holds for the ψ_2-norm of $\mathrm{W}_\omega(\mu_n, \nu_n)$ in place of the L^1-norm.

As another variant based on the application of (6), we also have

$$c\,\mathbb{E}\,(\mathrm{W}_\omega(\mu_n, \nu_n)) \le C_q \frac{\sqrt{d}}{\sqrt{n}} \bigg(\sum_{1 \le \|m\|_\infty \le T} q(|m|)\, \omega^2\Big(\frac{\pi}{|m|}\Big) \bigg)^{1/2} + \omega\Big(\frac{\sqrt{d}}{T}\Big),$$

which should replace the inequality (2.5) in [1]. Also, the weaker version (7) gives

$$c\,\mathbb{E}\,(\mathrm{W}_\omega(\mu_n, \nu_n)) \le \frac{\sqrt{d\, \log(2T\sqrt{d})}}{\sqrt{n}} \bigg(\sum_{1 \le \|m\|_\infty \le T} \omega^2\Big(\frac{\pi}{|m|}\Big) \bigg)^{1/2} + \omega\Big(\frac{\sqrt{d}}{T}\Big). \tag{9}$$

We now specialize Proposition 2.1 to the modulus of continuity $\omega(\delta) = \delta^\alpha$ with parameter $0 < \alpha < 1$, in which case the Kantorovich distance becomes the Zolotarev distance $\mathrm{W}_\omega = \zeta_\alpha$. Then the transport bounds (8)–(9) may easily be simplified by optimizing the right-hand sides over $t > 0$ and $T > 0$.

Let us start with dimension $d = 1$ and apply (8)–(9) which respectively yield

$$c\,\mathbb{E}\,(\zeta_\alpha(\mu_n, \nu_n)) \le C_q \frac{1}{\sqrt{n}} \bigg(\sum_{m=1}^{\infty} \frac{q(m)}{m^{2\alpha}}\, e^{-tm^2} \bigg)^{1/2} + t^{\alpha/2} \tag{10}$$

and

$$c\,\mathbb{E}\,(\zeta_\alpha(\mu_n, \nu_n)) \le \frac{\sqrt{\log T}}{\sqrt{n}} \bigg(\sum_{1 \le m \le T} \frac{1}{m^{2\alpha}} \bigg)^{1/2} + \frac{1}{T^\alpha} \tag{11}$$

with arbitrary $t > 0$ and $T \ge 1$.

If $\alpha > \frac{1}{2}$, one may choose $q(x) = \log^2(2x)$ in (10) and let $t \to 0$. In this range, the conclusion of Corollary 3.2 of [1] is not modified, with the standard rate

$$\mathbb{E}\,(\zeta_\alpha(\mu_n, \nu_n)) \le \frac{c_\alpha}{\sqrt{n}}$$

and constant $c_\alpha \sim \big(\int_1^\infty x^{-2\alpha} \log^2 x\, dx \big)^{1/2} = (2\alpha - 1)^{-3/2}$ (where the equivalence is understood within absolute positive factors)

When $\alpha \le \frac{1}{2}$, there are additional logarithmic factors. If $\alpha = \frac{1}{2}$, then choosing $T = n$ in (11), we obtain

$$\mathbb{E}\,(\zeta_\alpha(\mu_n, \nu_n)) \le c\,\frac{\log(2n)}{\sqrt{n}}. \tag{12}$$

If $\alpha < \frac{1}{2}$, a similar choice leads to

$$\mathbb{E}\,(\zeta_\alpha(\mu_n, \nu_n)) \le \frac{c_\alpha}{n^\alpha}\sqrt{\log(2n)}$$

for some constant $c_\alpha > 0$ depending on α only. All these bounds can be sharpened in terms of ψ_2-norms as discussed in [1].

The value $\alpha = \frac{1}{2}$ is therefore critical, in the sense that the rate in the upper bound is changing for smaller values of the parameter α. This threshold phenomenon was already emphasized in [1] in connection with Bernstein's theorem on the absolute convergence of Fourier series for Lipschitz classes $\mathrm{Lip}(\alpha)$.

If $d \ge 2$, the rates are different and depend on d. Namely, using (9) we get that, for all $0 < \alpha < 1$,

$$\mathbb{E}\,(\zeta_\alpha(\mu_n, \nu_n)) \le \frac{c_\alpha(d)}{n^{\alpha/d}}\sqrt{\log(2n)} \tag{13}$$

with some constants $c_\alpha(d)$ depending on α and d only. This bound should replace the one in Corollary 3.3 in [1]. It is interesting that (13) is optimal with respect to n for the critical value $\alpha = 1$ in dimension $d = 2$ and represents the contents of the AKT theorem [2] (however, this cannot be achieved on the basis of (9)).

3 Minimax Grid Matching

Let us recall that, for two collections of points $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ in the unit interval [0, 1], the minimax matching length is defined to be

$$\mathrm{L}(X, Y) = \min_\sigma \max_{1 \le k \le n} |x_k - y_{\sigma(k)}|,$$

where the minimum is running over all permutations σ of $\{1, \dots, n\}$.

If X and Y are independent samples drawn from a given distribution μ, the minimax grid matching problem is to find the rate of $\mathbb{E}\,(\mathrm{L}(X, Y))$ at which it tends to zero as $n \to \infty$. When μ is a uniform distribution, it was shown by T. Leighton and P. Shor that

$$\mathbb{E}\,\big(\mathrm{L}(X, Y)\big) \sim \frac{1}{\sqrt{n}}.$$

Using the corrected version of Corollary 3.2 of [1] in the form (12), one can sharpen this standard rate, if counting not all, but most of the points in perfect matching.

Namely, for a (non-empty) subset I of $\{1, \dots, n\}$, we defined the restricted minimax matching length

$$\mathrm{L}_I(X, Y) = \min_{\sigma} \max_{k \in I} |x_k - y_{\sigma(k)}|,$$

still assuming that the minimum is running over all permutations σ of $\{1, \dots, n\}$. Since the right-hand side of the inequality (12) has an additional factor $\sqrt{\log(2n)}$ in comparison to Corollary 3.2, a slight logarithmic correction is also needed in Proposition 4.1 of [1], replacing the original $\log^2(2n)$ by $\log^3(2n)$. As before, there is no need to keep the assumption that the distributions of the components X_k are identical.

Proposition 4.1 (Corrected version) *Let $Y = (Y_1, \dots, Y_n)$ be an independent copy of the random vector $X = (X_1, \dots, X_n)$ which has independent coordinates with values in* $[0, 1]$. *With high probability, for each $\varepsilon > 0$, there is a (random) set $I \subset \{1, \dots, n\}$ of cardinality $|I| \geq (1 - \varepsilon)\, n$ such that*

$$\mathrm{L}_I(X, Y) \leq C_\varepsilon \, \frac{\log^3(2n)}{n},$$

where one may take $C_\varepsilon = C/\varepsilon^2$ with an absolute constant C.

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